TWO SIMPLE APPROXIMATE METHODS OF LAPLACE TRANSFORM INVERSION FOR VISCOELASTIC STRESS ANALYSIS

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SUMMARY

Two approximate methods of Laplace transform inversion are given which are simple to use and are particularly applicable to stress analysis problems in quasi-static linear viscoelasticity. Once an associated elastic solution is known numerically or analytically, the time-dependent viscoelastic response can be easily calculated using realistic material properties, regardless of how complex the property dependence of the elastic solution may be. The new feature of these methods is that it is necessary to know only 1) an elastic solution numerically for certain ranges of elastic constants and 2) numerical values of the operational moduli or compliances for real, positive values of the transform parameter. One method utilizes a mathematical property of the Laplace transform, while the other is based on some results obtained from Irreversible Thermodynamics and variational principles. Because of this, they are quite general and can be used with anisotropic and inhomogeneous materials. Two numerical examples are given: As the first one, we calculate the time-dependent strain in a long, internally pressurized cylinder with an elastic case. The second example consists of inverting a transform which was derived by Muki and Sternberg in the thermo-viscoelastic analysis of a slab and a sphere[1]. Both methods were found to provide results which are within the usual engineering requirements of accuracy. Application of the approximate methods to problems in dynamic viscoelasticity is discussed briefly.

Supplementing the stress analysis, two techniques for calculating operational moduli and compliances from experimental stress-strain data are discussed and applied. Both can be used with creep, relaxation, and steady-state oscillation data. The most direct one consists of numerically integrating experimental data, while the other is a model-fitting scheme. With this latter method finite-element spring and dashpot models are readily found which fit the entire response curves. In using these methods to calculate the operational functions employed in the stress analysis examples, we found that model-fitting was the fastest of the two, yet was very accurate.

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1. INTRODUCTION

For a large class of problems in linear viscoelasticity the correspondence principle can be used to calculate a time-dependent viscoelastic response from the solution to an "associated" elastic problem. The basis for this rule is that, with zero initial conditions, the Laplace (or Fourier) transformed (with respect to time) viscoelastic field equations and boundary conditions are formally identical with the equations for an elastic body of the same geometry. Thus, transformed solutions can be calculated by standard elastic analysis, and then inverted to obtain the time-dependent response. This principle was deduced by Lee[2] for isotropic media and later by Biot[3] for anisotropic materials. Biot has indicated that the principle is applicable also to variational methods of approximate elastic analysis. Furthermore, Lee observed that with proportional loading (i.e. the space and time dependence of prescribed loads and displacements appear as separate factors with a common time factor) the spacial dependence of a transformed viscoelastic solution is the same as occurs in a geometrically similar elastic body if the spacial dependence of prescribed quantities is the same for both problems. A practical implication of this latter point is that, with proportional loading, a transformed solution can be derived directly from an elastic solution by replacing elastic constants by operational moduli (or compliances) and the time dependence of prescribed loading and displacements by transformed quantities.

Whether or not there is proportional loading, it is important to recognize that elastic analysis can be used to calculate only transformed solutions. The final step of inverting the transforms may often prove to be extremely difficult if standard exact or asymptotic methods are used. This has been found to be true, except for some elementary cases, particularly when realistic material properties are employed. An example of the complexity which arises with the use of actual material properties can be seen in the paper by Muki and Sternberg[1]. They found it necessary to use asymptotic methods of inversion, but even these were very laborious.

It should be noted that use of the Laplace (or Fourier) transform and correspondence principle is restricted to those problems in which application of the transform removes time dependence and replaces it by

algebraic dependence on the transform parameter. This limits it to problems in which boundaries do not move (other than infinitesimal deformation), the type of boundary condition (i.e. stress or displacement) at each surface point does not change with time, and the differential equations relating stress and strain have time-independent coefficients.

This restriction on the class of problems, as well as the difficulty encountered in the inversion, have motivated the development of an integral equation formulation of linear viscoelasticity. Elder[4] has derived some general equations, and Lee and Rogers[5] have shown that the use of integral equations often leads to a relatively simple numerical scheme for calculating the time dependence. In this formulation, experimentally measured creep or relaxation functions are used in hereditary (convolution-type) integrals as a means of relating stress and strain. Lee[5] gives an example in which the convolution-type integral equation that arises is solved approximately by a finite difference method of integration; he shows that the convolution property generates a triangular matrix which is readily inverted to provide a good approximate solution. It has also been indicated that with integral equations the types of boundary conditions which the Laplace transform cannot handle, as cited above, can be treated. While the use of integral equations seems to be a convenient means of solving many viscoelastic problems, it nevertheless will be quite laborious to employ in situations where material properties enter in a complicated fashion.

The Fourier transform has also been employed in order to conveniently utilize realistic material properties[6], however it too suffers from the fact that inversion is difficult whenever the associated elastic solution is a complicated function of material properties.

In this paper we go back to the use of the Laplace transform and correspondence principle, but remove the difficulty which has been associated with this technique by developing two simple, approximate methods of inversion. These will be seen to be particularly applicable to problems in which inertia effects (due to straining) are neglected, and prescribed loads (thermal and mechanical) and displacements are step functions of time. Of course, this latter limitation is not serious since once the response to a step function input has been calculated, it can be used to generate solutions to general time-dependent load and displacement

prescriptions by means of the superposition principle (convolution integral).

The first of the two methods derived in section two is based on a mathematical property of the Laplace transform, and is a modification of Alfrey's rule[7]. The second one is a least-squares technique which is based on some general results derived from Irreversible Thermodynamics and variational principles[8]. Because of the general bases for these methods, they can be used for materials with any degree of anisotropy and inhomogeneity. With both, it is necessary to know only an associated elastic solution numerically for certain ranges of elastic constants and numerical values of the operational moduli (or compliances) for real, positive values of the transform parameter. These operational quantities are related to one another through the same equations which connect the corresponding elastic constants. It will be seen that once these numerical quantities have been calculated the time dependent response is obtained with very little effort, regardless of how complex the property dependence of the associated elastic solution may be. The fact that an analytical representation of solutions is not needed has the important implication that elastic solutions generated on high speed computors can be used to calculate viscoelastic responses.

In section three we apply both methods to two relatively simple problems for which solutions have been previously obtained by other means. As the first example we calculate the tangential strain at the inner boundary of a hollow, thick-walled circular cylinder of polyisobutylene; it is assumed to be in plane strain and loaded internally by a suddenly applied, constant pressure, while the outer boundary is supported by a thin elastic shell. The operational shear compliance is used, and is calculated in the appendix from experimental values of the real part of the complex compliance. To illustrate possible ways of making this latter calculation, both a numerical integration procedure and a model fitting scheme are used.

For the second application, we invert a transform which was derived in a thermo-viscoelastic problem by Muki and Sternberg[1]. They used asymptotic expansions to obtain an approximate solution, while Lee and Rogers[5] later applied the integral equation method mentioned above. The relaxation tensile modulus for polymethyl methacrylate is used, and in the appendix we calculate the operational tensile modulus from this data; as in the first example, the alternate techniques of using either direct integration or models are considered.

2. APPROXIMATE METHODS OF LAPLACE TRANSFORM INVERSION

a. Direct Method.

Of the two inversion techniques to be discussed in this section the simplest one is called the "direct method", which we shall discuss first. It will be shown to yield good results whenever the derivative of the time dependent solution with respect to logarithmic time, log t, is a slowly varying function of log t. A modification of this method is also suggested in order to handle functions for which the derivative of the logarithm has this slowly varying property.

The problem which we pose is to find an approximate representation of a viscoelastic response, $\psi(t)$, from the integral equation*

$$\bar{\psi}(p) = \int_0^\infty \psi(t)\, e^{-pt}\, dt \qquad (2.1)$$

where $\bar{\psi}(p)$ is the Laplace transform of $\psi(t)$ and is known at least numerically for all real, non-negative values of the transform parameter, p. As a motivation for the change of variables which will be made in this integral, we make the following observations: If $\psi(t)$ is the solution to a problem in which all prescribed loads and displacements are step functions of time applied at $t = 0$**, the body is undisturbed for $t < 0$, and inertia is neglected, then $p\bar{\psi}(p)$ depends on p only through operational moduli (or compliances); this follows from the remark made early in the introduction since this is a case of proportional loading and the transforms of all prescribed quantities are proportional to $1/p$. Furthermore, $p\bar{\psi}(p)$ is identical with the solution for a geometrically similar elastic body, which is subjected to the same prescribed loads and displacements, if the elastic constants are replaced by corresponding operational moduli. Now, it is found that $d(p\bar{\psi}(p))/d(\log p)$ is a slowly varying function of log p ($\log \equiv \log_{10}$) for many viscoelastic problems when inputs are step-functions in time; this is particularly true for polymers due to the wide distribution of relaxation times which they have. Consequently, it is convenient to represent $p\bar{\psi}(p)$ as a function of log p and define

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*The functions considered in this section may be functions of the spacial coordinates, x_i ($i = 1, 2, 3$), however for simplicity of notation we shall omit explicit indication of this.

**A step function, H(t), is defined as: H(t) = 0 for $t < 0$; H(t) = constant with respect to time for $t > 0$.

$$\hat{f}(u) \equiv P\overline{\psi}(P) \tag{2.2}$$

where $u \equiv \log P$

We shall also define

$$f(v) \equiv \psi(t) \tag{2.3}$$

with $v \equiv \log t$

and make the corresponding variable change in (2.1), thus

$$\hat{f}(u) = \ln 10 \int_{-\infty}^{\infty} f(v)\, 10^{(u+v)}\, e^{-10^{(u+v)}}\, dv \tag{2.4}$$

Also, let the variable of integration be changed to α where $\alpha \equiv u + v$, which renders (2.4) in the form

$$\hat{f}(u) = \ln 10 \int_{-\infty}^{\infty} f(\alpha - u)\, 10^{\alpha}\, e^{-10^{\alpha}}\, d\alpha \tag{2.5}$$

The weighting function, $10^{\alpha} e^{-10^{\alpha}}$, is drawn in Figure 1 which shows that it is practically a delta function if f(v) changes slowly enough. This suggests that an approximation to f(v) can be obtained directly in terms of $\hat{f}(u)$ by replacing $(\ln 10) 10^{\alpha} e^{-10^{\alpha}}$ with $\delta(\alpha - \alpha_0)$, i.e. a Dirac delta function located at the point α_0*. This approximation yields a direct inversion formula

$$f(\alpha_0 - u) \simeq \hat{f}(u) \tag{2.6}$$

The point α_0 is somewhat arbitrary in view of the spread of the weighting function, which is about two decades. However we shall now calculate the "best" value to use when f is closely approximated by a straight line in the two-decade interval, $|\alpha - \alpha_0| \lesssim 1$. To do this we first expand f(v) in a Taylor series about the point V_0 $(\equiv \alpha_0 - u)$

$$f(v) = f(V_0) + f'(V_0)(v - V_0) + \tfrac{1}{2} f''(V_0)(v - V_0)^2 + \cdots \tag{2.7}$$

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*It should be recalled that a delta function is defined such that

$$\delta(\alpha - \alpha_0) = 0 \quad \text{if} \quad \alpha \neq \alpha_0$$

$$\int_{-\infty}^{\infty} \delta(\alpha - \alpha_0)\, d\alpha = 1$$

where primes denote differentiation with respect to v. Substitution of this expansion into (2.5) yields

$$\hat{f}(u) \simeq f(V_o) + (\ln 10) f'(V_o) \int_{-\infty}^{\infty} (\alpha - \alpha_o) 10^{\alpha} e^{-10^{\alpha}} d\alpha \qquad (2.8)$$

where only the constant and linear term in (2.7) have been retained. It is seen that the approximate inversion formula (2.6) is obtained if the integral in (2.8) vanishes. This condition locates α_o at the centroid of the area under the curve $10^{\alpha} e^{-10^{\alpha}}$, which is

$$\alpha_o = \int_0^{\infty} (\log t)\, e^{-t}\, dt = -\frac{C}{\ln 10} \qquad (2.9a)$$

where C is Euler's constant defined as

$$C \equiv -\int_0^{\infty} (\ln t)\, e^{-t}\, dt \simeq 0.58 \qquad (2.9b)$$

When this value for α_o is substituted into (2.6), and the resulting expression is written in terms of the original functions $\psi(t)$ and $p\bar{\psi}(p)$ by using (2.2) and (2.3), we find*

$$\left[\psi(t)\right]_{t=\frac{e^{-C}}{p}} \simeq p\bar{\psi}(p) \qquad (2.10a)$$

or equivalently

$$\psi(t) \simeq \left[p\bar{\psi}(p)\right]_{p=\frac{e^{-C}}{t}} \qquad (2.10b)$$

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*The fact that the weighting function is essentially zero except for a two decade interval suggests that this inversion will be good at those times in which the solution (as a function of log t) is linear, or near linear, for at least two decades. If there is a strong curvature, but it is somewhat removed from this linear region, this curvature should not produce a significant error in the linear portion.

where $e^{-c} \simeq 0.56$. Due to the skewed form of the weighting function it has been found that a somewhat better formula is*

$$\psi(t) \simeq \left[p \bar{\psi}(p) \right]_{p = \frac{0.5}{t}} \tag{2.10c}$$

It was mentioned in the introduction that this direct inversion method is related to Alfrey's rule[7]. He considered the problem of finding the distribution function $H(\tau)$ from the integral equation,

$$\Delta E_R(t) = \int_0^\infty H(\tau)\, e^{-\frac{t}{\tau}}\, d\tau \tag{2.11}$$

where $\Delta E_R(t)$ is the time-dependent component of the relaxation modulus, which is known numerically for $0 \le t < \infty$. By differentiating $\Delta E_R(t)$ and changing the integration variable this integral can be converted to the Laplace transform of $H(\frac{1}{\lambda})$:

$$\frac{d\,\Delta E_R}{dt} = -\int_0^\infty H(\tfrac{1}{\lambda})\, e^{-\lambda t}\, d\lambda \tag{2.12}$$

where t is now the transform parameter. Alfrey stated that an approximate solution is given by

$$H(\tau) \simeq -\left(t\, \frac{d\,\Delta E_R}{dt} \right)_{t=\tau} \tag{2.13}$$

which can be derived by replacing the weighting function, Figure 1, with a delta function located at $\alpha = 0$. In contrast, the direct method discussed

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*In the event that $\log \left[p \bar{\psi}(p) \right]$ is essentially linear with a slope of n when plotted against $\log p$, so that $p \bar{\psi}(p) \simeq A p^n$, it can be easily verified for $n < 1$ that:

$$\psi(t) \simeq \left[p \bar{\psi}(p) \right]_{p = \frac{\beta}{t}}$$

where

$$\beta = \left[\Gamma(1-n) \right]^{-\frac{1}{n}}$$

which takes the values

$$\begin{aligned} \beta &> 1 \quad \text{for} \quad n < -1 \\ \beta &= 1 \quad \text{for} \quad n = -1 \\ e^{-c} &< \beta < 1 \quad \text{for} \quad -1 < n < 0 \\ \beta &= e^{-c} \quad \text{for} \quad n = 0 \\ 0 &< \beta < e^{-c} \quad \text{for} \quad 0 < n < 1 \end{aligned}$$

However the approximation is poor when n is close to +1.

above suggests that a better approximation is*

$$H(\tau) \simeq -\left(t \frac{d\Delta E_R}{dt}\right)_{t=0.5\tau} \tag{2.14a}$$

or, as the case may be (see footnote, page 7)

$$H(\tau) \simeq -\left(t \frac{d\Delta E_R}{dt}\right)_{t=\beta\tau} \tag{2.14b}$$

As another point, it is recalled that $p\bar{\psi}(p)$ can be interpreted as the solution to an elastic problem (with appropriate values for the elastic constants). Consequently, the inversion formula (2.10c) states that the viscoelastic and elastic solutions are practically equal whenever $d\psi/d(\log t)$ is a slowly varying function of log t. This correspondence has an important implication in regard to approximate solutions obtained from variational principles, since in this case the "quality" of a viscoelastic approximation is essentially the same as that of the elastic problem.

Finally, we should remark that there does not appear to be an easy method of estimating, quantitatively, the error involved in this direct method. Consequently, it is necessary to assume that if the transform of the approximate inversion and the original transform are close for $p \geq 0$, the error in time dependence is small. This would seem to be a valid assumption if the time dependence predicted by (2.10c) is physically reasonable. Another check on this approximation can be made by comparing it with the results of the method to be discussed next.

b. Least-squares method (collocation)

We would now like to discuss a technique which is not as simple as the direct method, but it has other advantages. For one, it is not restricted to functions whose derivative is slowly varying with respect to logarithmic time. In addition, the time dependence is given by a simple series of exponentials which can be used readily in the Duhamel superposition integral for the calculation of responses to prescribed loads and displacements that are not step functions. Thirdly, the accuracy of the

*Other improvements on Alfrey's rule have been used[7], but they require evaluation of higher order derivatives of $\Delta E_R(t)$.

inversion can be improved by adding more terms to the series.

Let us first give some results derived in another study in order to provide a motivation for the second method[8]. We have used Irreversible Thermodynamics and variational principles to predict the time-dependent response of initially undisturbed viscoelastic bodies subjected to prescribed loads (thermal and mechanical) and displacements which are step functions of time applied at t = 0. In the study inertia due to straining was neglected. Denoting a response (stress or displacement) by $\psi(t)$, which may be either an exact solution or else an approximate one obtained from a variational principle, it was argued that the most general behavior of $\psi(t)$ is

$$\psi(t) = \Delta\psi(t) + \psi + \psi' t \tag{2.15a}$$

where ψ and ψ' are constants with respect to time and $\Delta\psi(t)$ is the transient component given by the integral:

$$\Delta\psi(t) = \int_0^\infty \Phi(\tau)\, e^{-\frac{t}{\tau}}\, d\tau \tag{2.15b}$$

The function $\Phi(\tau)$ is called a spectral distribution function of the variable τ, and may consist entirely or partly of Dirac delta functions. For example, if $\Phi(\tau)$ is a sum of delta functions,

$$\Phi(\tau) = \sum_{\alpha=1}^{m} \Phi_\alpha\, \delta(\tau - \tau_\alpha) \tag{2.16}$$

then

$$\Delta\psi(t) = \sum_{\alpha=1}^{m} \Phi_\alpha\, e^{-\frac{t}{\tau_\alpha}} \tag{2.17}$$

Furthermore, either ψ or ψ' may vanish depending on the problem; if, for example, $\psi(t)$ corresponds to a stress or the body does not exhibit steady-flow then ψ' is zero. An important implication of (2.15) is that the Laplace transform of a viscoelastic stress or strain has singularities only on the non-positive real p-axis, and that all poles are simple except at the origin where a double pole may occur.

It should be added that the behavior (2.15) was derived for only those problems in which the Laplace transform is applicable*. In addition, we gave a rigorous proof for just the case in which the response of a body can be defined by a finite number of generalized coordinates (or degrees of freedom). For an infinite number of generalized coordinates, the dependence (2.15) was postulated as a natural extension of the finite degree of freedom system response. Some numerical examples were given to strengthen this postulate. It should be emphasized that because of the general thermodynamic basis for the postulate, it is applicable to anisotropic and inhomogeneous materials.

The time dependence of the exact inversion suggests that a Dirichlet series

$$\Delta\psi_D = \sum_{i=1}^{n} S_i e^{-\frac{t}{\gamma_i}} \tag{2.18}$$

can be used as a reasonable approximation to the solution $\Delta\psi(t)$. The present method makes use of this series for which the γ_i are prescribed positive constants, and the S_i are unspecified coefficients to be calculated by minimizing the total square error between $\Delta\psi$ and $\Delta\psi_D$.

The total square error is

$$E^2 = \int_0^\infty [\Delta\psi - \Delta\psi_D]^2 \, dt \tag{2.19}$$

with the minimization yielding

$$-\frac{1}{2}\frac{\partial E^2}{\partial S_i} = 0 = \int_0^\infty [\Delta\psi - \Delta\psi_D] e^{-\frac{t}{\gamma_i}} dt \quad (i=1,2,\cdots n) \tag{2.20a}$$

so that n relations are obtained between the Laplace transforms of $\Delta\psi$ and $\Delta\psi_D$ evaluated at $1/\gamma_i$,

$$\overline{\Delta\psi_D}\left(\frac{1}{\gamma_i}\right) = \overline{\Delta\psi}\left(\frac{1}{\gamma_i}\right) \qquad (i=1,2,\cdots n) \tag{2.20b}$$

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*Thus, if a transient temperature exists in a body and properties are temperature dependent, the behavior (2.15) does not necessarily occur. If, however, the time-temperature superposition principle[1] is applicable and a change of the time variable can be made which removes the time dependence of coefficients, then (2.15) applies to the "reduced time" variable.

A more convenient form is obtained by multiplying these by γ_i , which yields

$$\left[P\Delta\,\overline{\psi_D}(P)\right]_{P=\frac{1}{\gamma_i}} = \left[P\Delta\overline{\psi}(P)\right]_{P=\frac{1}{\gamma_i}} \tag{2.21a}$$

or, explicitly

$$\sum_{j=1}^{n} \frac{S_j}{1+\frac{\gamma_i}{\gamma_j}} = \left[P\Delta\overline{\psi}(P)\right]_{P=\frac{1}{\gamma_i}} \qquad (i=1,2,\cdots n) \tag{2.21b}$$

These equations are sufficient for calculation of the approximate time dependence of the transient component of $\psi(t)$. To obtain the total solution, whose most general form is

$$\psi_D(t) = \Delta\psi_D(t) + \psi + \psi' t \tag{2.22}$$

we must evaluate the constants ψ and ψ'. However, these can be readily obtained by exact means through an examination of the behavior of $P\overline{\psi}(P)$ and $P^2\overline{\psi}(P)$ as p tends to zero.

Thus, we see that the total square error is minimized by collocating the p-multiplied transform of the Dirichlet series (2.18) and an elastic solution, $P\overline{\psi}(P)$, at n points $P=1/\gamma_i$. With this elastic solution known at least numerically or graphically for $0 \le p < \infty$, suitable values of γ_i can be prescribed simply by inspection.

With regard to the error involved in this approximate method, it is of interest to calculate the total square error, E^2, by using (2.15b) and the relations (2.20b). We have

$$E^2 = \int_0^\infty [\Delta\psi - \Delta\psi_D]\left[\int_0^\infty \phi(\tau) e^{-\frac{t}{\tau}} d\tau - \sum_{i=1}^{n} S_i e^{-\frac{t}{\gamma_i}}\right] dt \tag{2.23}$$

which becomes

$$E^2 = \int_0^\infty \phi(\tau)\left[\Delta\overline{\psi}\left(\tfrac{1}{\tau}\right) - \Delta\overline{\psi_D}\left(\tfrac{1}{\tau}\right)\right] d\tau \tag{2.24}$$

where it is necessary to assume $\phi(\tau)$ is such that the order of integration in (2.23) can be interchanged[9] and that the final integral (2.24) is

convergent. If these reasonable assumptions are valid, then eqn. (2.24) shows that when $\Delta\overline{\psi}(p)$ and $\Delta\overline{\psi}_D(p)$ are "close" (which can be readily ascertained from the elastic solution) the square error of the approximate inversion, $\Delta\psi_D(t)$, is "small".

Finally, a question arises as to whether the two approximate methods can be used successfully for problems with inertia, such as wave propagation. Considering the direct method, it is expected that it can be used in some cases to obtain at least part of a solution. If there is a sufficiently broad time interval where, for physical reasons, the "slow variation" property is anticipated, one could determine this portion of the response by the direct method. By combining this result with asymptotic expansions (at wave fronts, for example), a reasonable approximation to the total response might be obtained. Similarly, since the collocation method is equivalent to a least-squares procedure in the time dimension, regardless of the form of time dependence of the exact inversion, it may yield good results if used with care. However, this dynamic area of application needs further study before any definite conclusions can be reached.

3. NUMERICAL EXAMPLES

a. Pressurization Problem

For the first application of the two methods of approximate inversion we shall calculate the tangential strain at the inner boundary of a hollow, thick-walled circular cylinder which is made of an isotropic, homogeneous, viscoelastic material. The cylinder is assumed to be in plane strain and loaded internally by a unit Heaviside step-pressure, H(t), while the outer boundary is supported by a thin elastic shell. Since the associated elastic solution can be easily obtained by exact methods, and is derived in reference 6, we shall not give the algebraic details here. Rather, only an expression for the strain will be given which was obtained from the exact solution by assuming certain values for properties of the elastic shell and geometrical parameters, and by making some reasonable approximations in order to simplify the result. Under these assumptions, the elastic strain was found to be[6]

$$\varepsilon = \frac{KJ + 1.33}{KJ + 76.82} H(t) \tag{3.1}$$

where

K = bulk modulus

J = shear compliance (reciprocal of the shear modulus) and ξ is the tangential strain which has been normalized to unity for $J \to \infty$.

The elastic solution (3.1) is formally identical with the p-multiplied transform of the viscoelastic strain, thus

$$p\bar{\xi}(p) = \frac{K(p)\,J(p) + 1.33}{K(p)\,J(p) + 76.82} \tag{3.2}$$

where $J(p)$ is the operational shear compliance which is defined as the ratio of transformed shear strain, $\bar{\gamma}$, to transformed shear stress, $\bar{\tau}$,

$$J(p) \equiv \frac{\bar{\gamma}}{\bar{\tau}} \tag{3.3}$$

and the operational bulk modulus, $K(p)$, is defined in a similar manner.

In reference 6, equation (3.2) was used with the Fourier transform to calculate the strain for glass-filled polyisobutylene at 12.5°C which was assumed to be elastic in bulk ($K = 13.8 \times 10^{10}$ dynes/cm^2). The Fourier transform was employed in order that experimental shear data, obtained under constant frequency sinusoidal loading, could be used directly in numerical evaluation of the inversion integral. (Such a test provides the frequency dependence of the so-called complex shear compliance, which is equal to the ratio of the Fourier transformed shear strain to the transformed stress.) The viscoelastic strain is shown in Figure 2 and is labeled "exact solution".

Let us now apply the two approximate methods by using (3.2) with p real. Since $\xi(t)$ is already known for our simple example, it would be possible to calculate $p\bar{\xi}(p)$ directly by numerical integration. However, for illustrative purposes, we shall indicate the steps which are necessary for the calculation of $p\bar{\xi}(p)$ in the usual case when the exact solution is not known. Since the bulk modulus, K, is assumed constant, the first step is to determine J(p) from the available complex compliance data. This is carried out in the appendix and the result is plotted in Figure A-1.

Direct method: The time dependence prediction by the direct method is easily obtained by substituting numerical values of J(p) into (3.2) and replacing p by $0.5/t$. The resulting curve is shown in Figure 2, and is seen to be in good agreement with the exact dependence.

It is interesting to observe that if the relationship $t = 1/p$ had been used instead of $t = 0.5/p$, the approximate response would not be very good since it would be shifted rigidly 0.3 logarithmic units to the right of the illustrated curve. Another point of interest concerns the fact that the approximate solution lies above the exact one where the dependence is concave upward, with just the opposite being true when the solutions are concave downward. This relationship can be predicted just by examination of (2.5) and (2.7) since the difference between the two curves results from the quadratic and higher order terms in (2.7); these terms represent the difference between f(v) and a straight line which is tangent at v_o.

Least-squares method: For the least-squares method we assume an approximate solution in the form

$$\xi_D = S_o + \sum_{i=1}^{n} S_i e^{-\frac{t}{\gamma_i}} = S_o + \Delta\xi_D(t) \tag{3.4}$$

where

$$S_o = \left[p\,\bar{\xi}(p)\right]_{p=0} = 1$$

and the coefficients, S_i, are calculated from the system of equations (2.21b) with $p\Delta\bar{\xi}(p)$ replacing $p\Delta\bar{\psi}(p)$. Examination of $p\,\bar{\xi}(p)$ in Figure 2 suggests that a five-term series ($n = 5$) will provide a good approximation with

$$\gamma_i = 10^{-(2+i)} \qquad (i = 1, 2, \cdots 5) \tag{3.5}$$

Substitution of these time constants and values for $\left[p\,\Delta\bar{\xi}(p)\right]_{p=\frac{1}{\gamma_i}}$ into (2.21b) yields the following system of equations:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{11} & \frac{1}{101} & \frac{1}{1001} & \frac{1}{10001} \\ \frac{1}{1.1} & \frac{1}{2} & \frac{1}{11} & \frac{1}{101} & \frac{1}{1001} \\ \frac{1}{1.01} & \frac{1}{1.1} & \frac{1}{2} & \frac{1}{11} & \frac{1}{101} \\ \frac{1}{1.001} & \frac{1}{1.01} & \frac{1}{1.1} & \frac{1}{2} & \frac{1}{11} \\ \frac{1}{1.0001} & \frac{1}{1.001} & \frac{1}{1.01} & \frac{1}{1.1} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{pmatrix} = \begin{pmatrix} -0.034 \\ -0.122 \\ -0.337 \\ -0.627 \\ -0.823 \end{pmatrix} \tag{3.6}$$

The relative magnitude of the elements in the matrix permits a first estimation of the S_i to be made by solving the system:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{11} & 0 & 0 & 0 \\ \frac{1}{1.1} & \frac{1}{2} & \frac{1}{11} & 0 & 0 \\ 1 & \frac{1}{1.1} & \frac{1}{2} & \frac{1}{11} & 0 \\ 1 & 1 & \frac{1}{1.1} & \frac{1}{2} & \frac{1}{11} \\ 1 & 1 & 1 & \frac{1}{1.1} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{pmatrix} = \begin{pmatrix} -0.034 \\ -0.122 \\ -0.337 \\ -0.627 \\ -0.823 \end{pmatrix} \tag{3.7}$$

which is seen to be a simple set of equations. If this first approximation is denoted by $S_i^{(1)}$, then the second approximation, $S_i^{(2)}$, can be determined from the equations,

$$
\begin{aligned}
&\frac{S_1^{(2)}}{2} + \frac{S_2^{(2)}}{11} + \frac{S_3^{(1)}}{101} + \frac{S_4^{(1)}}{1000} + 0 = -0.034 \\
&\frac{S_1^{(2)}}{1.1} + \frac{S_2^{(2)}}{2} + \frac{S_3^{(2)}}{11} + \frac{S_4^{(1)}}{101} + \frac{S_5^{(1)}}{1000} = -0.122 \\
&\frac{S_1^{(2)}}{1.01} + \frac{S_2^{(2)}}{1.1} + \frac{S_3^{(2)}}{2} + \frac{S_4^{(2)}}{11} + \frac{S_5^{(1)}}{101} = -0.337 \\
&S_1^{(2)} + \frac{S_2^{(2)}}{1.01} + \frac{S_3^{(2)}}{1.1} + \frac{S_4^{(2)}}{2} + \frac{S_5^{(2)}}{11} = -0.627 \\
&S_1^{(2)} + S_2^{(2)} + \frac{S_3^{(2)}}{1.01} + \frac{S_4^{(2)}}{1.1} + \frac{S_5^{(2)}}{2} = -0.823
\end{aligned}
\qquad (3.8)
$$

Again we obtain a relatively simple set of equations. This second approximation was found to be sufficiently accurate since it satisfied the equations to the third decimal place. The values of S_i so calculated yield the time dependence

$$\xi_D = 1 - \left[0.053 e^{-10^3 t} + 0.080 e^{-10^4 t} + 0.368 e^{-10^5 t} + 0.302 e^{-10^6 t} + 0.103 e^{-10^7 t}\right] \qquad (3.9)$$

This solution is plotted in Figure 2, where it is seen that it departs from the exact solution only at very small times; however this error could be essentially removed by adding another term in the series and requiring that $\left[p\bar{\xi}_D(p)\right]_{p\to\infty} = \left[p\bar{\xi}(p)\right]_{p\to\infty}$

b) Thermo-Viscoelastic Problem

We now consider the inversion of a transformed solution which arose in the thermo-viscoelastic analysis of a slab and a sphere[1]. The function we shall deal with here does not represent a total solution, but must be combined with a prescribed function of temperature in a convolution integral in order to generate a stress. In view of our present interest we shall not reproduce the derivation of the transform nor discuss this latter straightforward calculation. With respect to the physics of the problem, we only remark that the existence of a transient temperature gave rise to time-dependent coefficients in the stress-strain equation. However, the time-temperature superposition principle was utilized in conjunction with a change of time variable in order to remove this difficulty, and thereby permit application of the Laplace transform with respect to the new "reduced time" variable.

The function to be inverted is (see equation (6), reference 5):

$$\overline{R}(p) = \frac{2}{3} E \left[\frac{\overline{E}_R(p) / E}{1 + (1-2\nu)\, p\, \overline{E}_R(p)/E} \right] \tag{3.10}$$

where $\overline{E}_R(p)$ is the Laplace transform of the tensile relaxation modulus $E_R(t)$,

$$\overline{E}_R(p) = \int_0^\infty E_R(t)\, e^{-pt}\, dt \tag{3.11}$$

and since it is of no consequence in our analysis we have retained the symbols p and t, however they should be interpreted as "reduced variables." Also, E is the initial (glassy) value of relaxation modulus and ν is the corresponding initial value of Poisson's ratio in a relaxation test. It is convenient for our purposes to express $\overline{R}(p)$ as a function of the operational modulus, $E(p)$, instead of $\overline{E}_R(p)$. This is easily done since (see appendix)

$$E(p) = p\, \overline{E}_R(p) \tag{3.12}$$

which, when substituted into (3.10) yields the p-multiplied transform of $R(t)$ in terms of the operational modulus,

$$p\, \overline{R}(p) = \frac{2}{3} E \left[\frac{E(p)/E}{1 + (1-2\nu)\, E(p)/E} \right] \tag{3.13}$$

For the specification of $E(p)$ and ν we shall use the relaxation modulus for polymethyl methacrylate, which is reproduced in Figure A-3 of the appendix, and let $\nu = 0.35$. $E(p)$ is calculated in the appendix and is shown in the same figure.

Direct Method: An approximate solution is given immediately by evaluating $p\, \overline{R}(p)$ and setting $p = \frac{0.5}{t}$. This approximation is plotted in Figure 3 along with the solution derived by Lee and Rogers from an integral equation.

Least-squares method: The approximate solution is taken in the form

$$\frac{R_D(t)}{E} = S_0 + \sum_{i=1}^{6} S_i\, e^{-\frac{t}{\gamma_i}} \tag{3.14}$$

where

$$S_0 = \left[\frac{P\bar{R}(P)}{E}\right]_{P=0} \simeq 0$$

and, as in the previous example, the coefficients, S_i, are evaluated using the system (2.21b) with $P\bar{\psi}(P)$ replaced by $P\bar{R}(P)/E$. However, we shall modify this system somewhat in order to make $R_D(0) = R(0)$, or, equivalently, $\left[P\bar{R}_D(P)\right]_{P\to\infty} = \left[P\bar{R}(P)\right]_{P\to\infty}$. Reference to $P\bar{R}(P)/E$ in Figure 3 suggests that we can take

$$\gamma_i = 10^{-(i-4)} \qquad (i = 1, 2, \cdots 6) \tag{3.15}$$

and collocate at the five values $P = \frac{1}{\gamma_i}$ $(i=1,\cdots 5)$ with the sixth condition being $\left[P\bar{R}_D(P)/E\right]_{P\to\infty} = 0.513$. The resulting set of equations for the S_i becomes

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{11} & \frac{1}{101} & \frac{1}{1001} & \frac{1}{10001} & \frac{1}{100001} \\ \frac{1}{1.1} & \frac{1}{2} & \frac{1}{11} & \frac{1}{101} & \frac{1}{1001} & \frac{1}{10001} \\ \frac{1}{1.01} & \frac{1}{1.1} & \frac{1}{2} & \frac{1}{11} & \frac{1}{101} & \frac{1}{1001} \\ \frac{1}{1.001} & \frac{1}{1.01} & \frac{1}{1.1} & \frac{1}{2} & \frac{1}{11} & \frac{1}{101} \\ \frac{1}{1.0001} & \frac{1}{1.001} & \frac{1}{1.01} & \frac{1}{1.1} & \frac{1}{2} & \frac{1}{11} \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} 0.070 \\ 0.160 \\ 0.283 \\ 0.392 \\ 0.461 \\ 0.513 \end{bmatrix} \tag{3.16}$$

In view of the similarity between the matrices in (3.16) and (3.6) the same iteration scheme is used to find

$$\frac{R_D(t)}{E} = 0.130\, e^{-10^{-3}t} + 0.039\, e^{-10^{-2}t} + 0.236\, e^{-10^{-1}t} - 0.008\, e^{-t} + 0.129\, e^{-10t} - 0.013\, e^{-10^{2}t} \tag{3.17}$$

which is plotted in Figure 3. It is observed that at short times this solution fluctuates about the solution of Lee and Rogers, and at long times ($t > 10^{3}$) it goes to zero more rapidly than the other solutions. The fluctuation behavior can be attributed to the fact that exponential terms in a Dirichlet series do not blend together smoothly if there is a separation of more than one decade in the arguments of any two neighboring exponentials. Because of the relatively small coefficient of e^{-t} there is essentially a two-decade difference between successive terms; hence, the resulting function R(t) is not very smooth in the vicinity of $t \approx 10^{-1}$ and 1. By adding one or more terms to the series with values of γ in this range, this behavior could be removed. Similarly, the solution at long times could be improved by including additional terms with $\gamma > 10^{+3}$.

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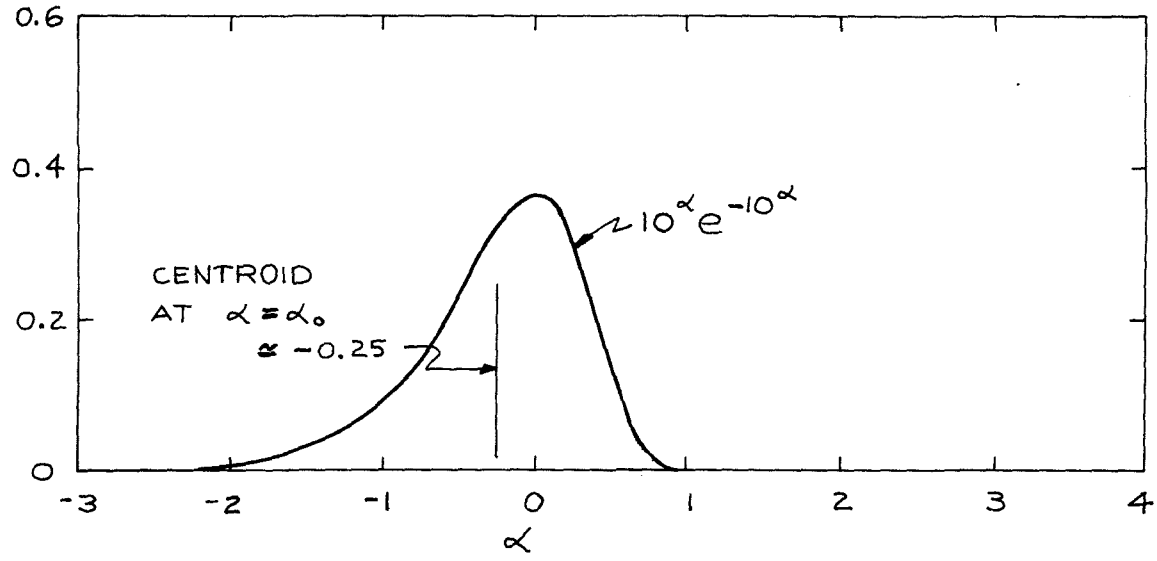

FIGURE 1. WEIGHTING FUNCTION IN LAPLACE TRANSFORM WITH LOGARITHMIC TIME SCALE

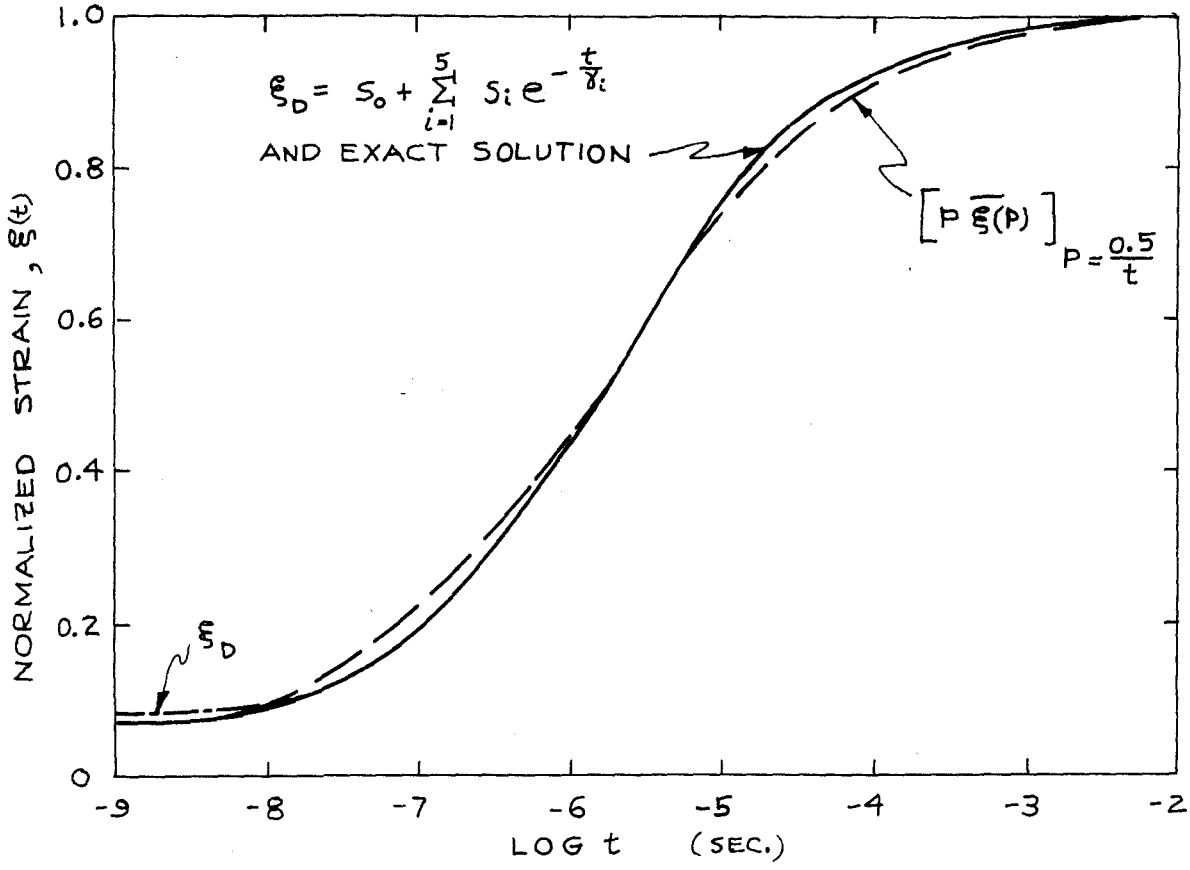

FIGURE 2. APPROXIMATE AND EXACT TIME DEPENDENCE OF STRAIN IN CYLINDER WITH POLYISOBUTYLENE

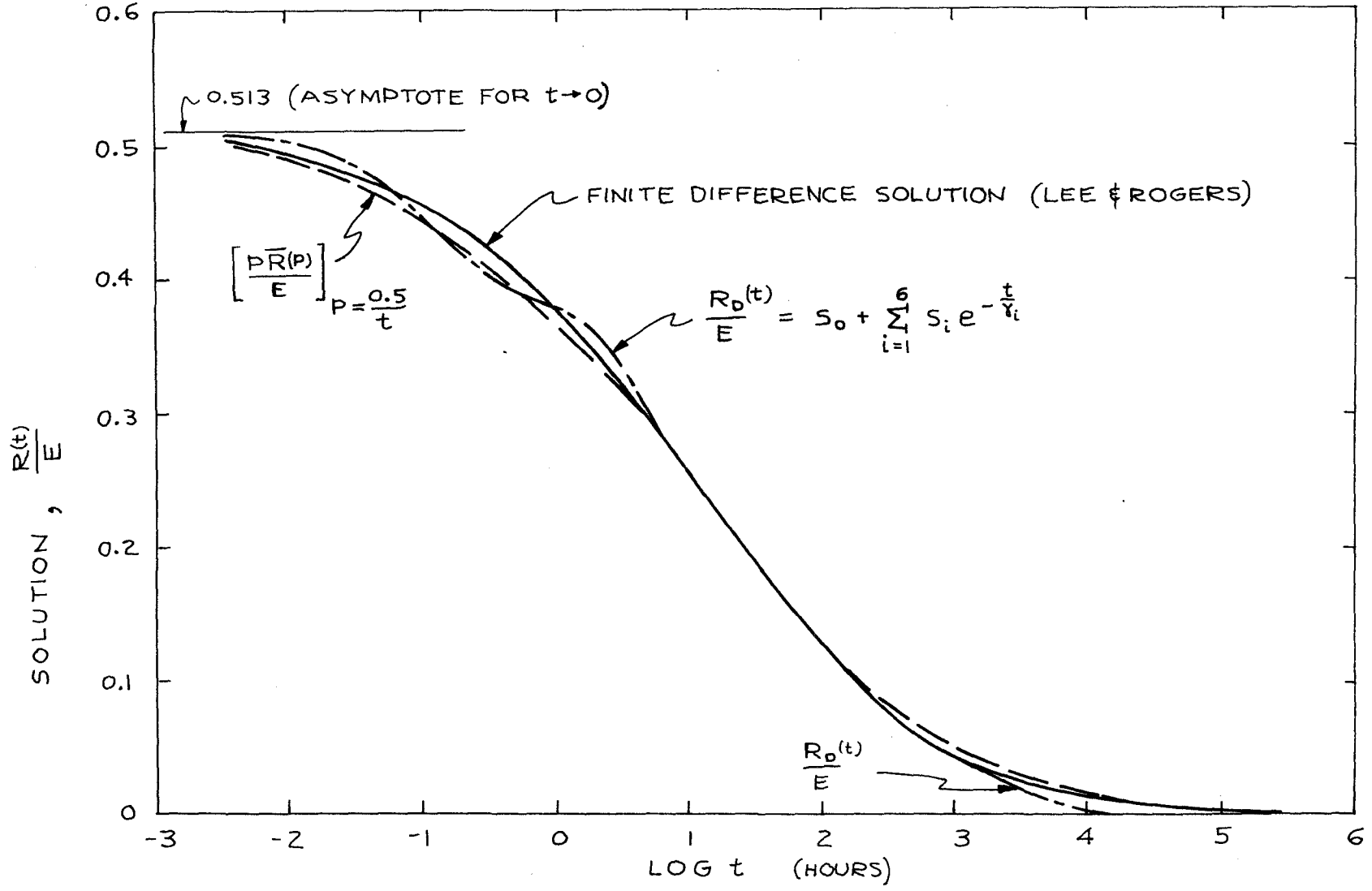

FIGURE 3. APPROXIMATE SOLUTIONS OF THERMO-VISCOELASTIC PROBLEM WITH POLYMETHYL METHACRYLATE

APPENDIX

We shall consider two methods of calculating operational moduli (or compliances) from experimental stress-strain data. The most direct one is to express the operational function as an integral of an experimentally measured function and then use numerical integration. However, because the measured response often varies over many decades of time, this method is more laborious than the second one presented, which is a collocation scheme for fitting the response of finite-element spring and dashpot models to experimental data. With this latter method the coefficients in a finite series representation of an operational modulus or compliance are obtained. In addition to being a more rapid procedure, model-fitting provides more information since an analytical approximation to the stress-strain law is found. This representation can be used with exact techniques of viscoelastic stress analysis. It also permits one to check the assumption of linearity by comparing the model's response to data other than that used in fitting the model (e.g. predict creep response from relaxation modulus). We should point out that the model-fitting method of collocation used here is different from other ones which have been suggested[10, 11, 12]. It possesses the advantages that a finite element model is easily found which closely fits the entire viscoelastic response curve, and can be applied to data from all of the standard tests, such as creep, relaxation, and steady-state oscillation.

Both methods will now be illustrated by calculating the operational functions used in the body of this report.

a. Property Representation of Polyisobutylene

Consider the problem of calculating the operational shear compliance, J(p), from the stress-strain data shown in Figure A-1, which was obtained on polyisobutylene under constant-frequency sinusoidal loading. The stress, $\sigma_0 e^{i\omega t}$, and strain, $\epsilon_0 e^{i\omega t}$, in such a test are related through the complex compliance $J^*(\omega)$[6] (ω = frequency),

$$\frac{\epsilon_0}{\sigma_0} = J^*(\omega) = \left[J(p)\right]_{p=i\omega} = J'(\omega) - i\,J''(\omega) \qquad (A.1)$$

where $J'(\omega)$ and $J''(\omega)$ are real functions representing the real and imaginary components of $J^*(\omega)$, respectively.

J(p) will now be expressed as an integral of $J'(\omega)$. The integral is derived by first writing the creep compliance as a Fourier sine transform of $J'(\omega)$ [13], and then taking the p-multiplied Laplace transform of the creep compliance. Omitting the details of this calculation we find

$$J(P) = J_g + \frac{2P}{\pi}\int_0^\infty \frac{[J'(\omega) - J_g]}{P^2+\omega^2}\,d\omega \;;\quad \text{Real } P \geq 0 \qquad (A.2)$$

where

$$J_g \equiv \left[J(\omega)\right]_{\omega\to\infty}$$

This integral was evaluated numerically and the result is shown in Figure A-1*.

Let us now apply the collocation method to determine J(p). With the help of model theory, the complex compliance is approximated by a finite sum[6], thus

$$J_s'(\omega) = J_g + \sum_{i=1}^{n}\frac{J_i}{\omega^2\tau_i^2+1} \qquad (A.3a)$$

and

$$J_s''(\omega) = \sum_{i=1}^{n}\frac{\omega\tau_i J_i}{\omega^2\tau_i^2+1} \qquad (A.3b)$$

In order to evaluate the model constants J_i and τ_i we collocate (A.3a) with experimental values at n points ω_i, and choose the τ_i such that $\omega_i\tau_i = 1$ $(i=1,2,\cdots n)$. Reference to Figure A-1 suggests that a good

---

*The similarity between the curves is a result of the fact that for several decades of frequency $J'(\omega)$ is essentially linear on the log-log plot. This similarity suggests that an approximation to J(p) is given by

$$J(P) \simeq \left[J'(\omega)\right]_{\omega=\alpha P}$$

where α is to be found by assuming $J' = A\omega^m$ and substituting this power law into (A.2). Such a procedure yields α as a function of the log-log slope, m.

fit will be obtained if

$$\omega_i = 10^{(i-2)} \qquad (i = 1, 2, \cdots 10) \tag{A.4a}$$

so that the "retardation times" τ_i are chosen to be

$$\tau_i = 10^{(2-i)} \qquad (i = 1, 2, \cdots 10) \tag{A.4b}$$

Collocation at the values (A.4a) yields the following system of linear equations for evaluation of the J_i *:

$$\begin{bmatrix}
\frac{1}{2} & \frac{1}{1.01} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2} & \frac{1}{1.01} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10^4+1} & \frac{1}{101} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \\ J_6 \\ J_7 \\ J_8 \\ J_9 \\ J_{10} \end{bmatrix}
=
\begin{bmatrix}
J'(10^{-1}) - J_g \\
J'(1) - J_g \\
J'(10) - J_g \\
J'(10^2) - J_g \\
J'(10^3) - J_g \\
J'(10^4) - J_g \\
J'(10^5) - J_g \\
J'(10^6) - J_g \\
J'(10^7) - J_g \\
J'(10^8) - J_g
\end{bmatrix} \tag{A.5}$$

---

*It may often be desirable to add another term to the series (A.3) in order to satisfy the condition

$$J_g + \sum_{i=1}^{n} J_i = \left[J'(\omega) \right]_{\omega = 0}$$

However for our example it turned out that this condition was closely satisfied without making it an explicit requirement.

The elements of the matrix which are shown as unity or zero were not exactly these values, but were close enough to permit the approximation. The matrix is nearly triangular and is similar to the one in (3.6). This type of matrix results from the fact that each term in the series (A.3a) is weakly dependent on frequency except for about 1 1/2 decades in the neighborhood of $\omega \sim 1/\tau_i$. Because of the relative magnitudes of the coefficients in the matrix we were able to use an iteration scheme similar to the one discussed previously for (3.6), which yielded the coefficients:

$$
\begin{aligned}
J_1 &= 357 \times 10^{-11} & J_6 &= 250 \times 10^{-11} & J_g &= 3.16 \times 10^{-11} \\
J_2 &= 533 \times 10^{-11} & J_7 &= 80.8 \times 10^{-11} & & \\
J_3 &= 3960 \times 10^{-11} & J_8 &= 22.2 \times 10^{-11} & & \\
J_4 &= 3580 \times 10^{-11} & J_9 &= 4.00 \times 10^{-11} & & \\
J_5 &= 1210 \times 10^{-11} & J_{10} &= 2.22 \times 10^{-11} & &
\end{aligned}
\qquad (A.6)
$$

The operational compliance is now immediately obtained since[6]

$$J(P) = J_g + \sum_{i=1}^{10} \frac{J_i}{\tau_i P + 1} \qquad (A.7)$$

It was found that a plot of (A.7) for $0 \leq p < \infty$ was graphically indistinguishable from the operational compliance calculated using (A.2).

It is interesting to observe from (A.3b) that we can predict the imaginary component of the complex compliance. The predicted and experimental curves are shown in Figure A-2; the close agreement, except at high frequencies, is a check on the assumption of linearity.

b. Property Representation of Polymethyl Methacrylate.

The experimental data which will be used is the tensile relaxation modulus defined as

$$E_R(t) \equiv \frac{\sigma_R(t)}{\epsilon_0} \qquad (A.8)$$

where $\sigma_R(t)$ is the uniaxial tensile stress and ϵ_0 is the constant strain which is applied stepwise at $t = 0$. The operational tensile modulus is simply related to $\overline{E}_R(P)$ since, in general

$$E(p) = \frac{\bar{\sigma}}{\bar{\epsilon}} \tag{A.9}$$

and with a relaxation test $\bar{\epsilon} = \epsilon_o/p$, so that

$$E(p) = \frac{p\bar{\sigma}}{\epsilon_o} \tag{A.10}$$

Comparing (A.10) with the Laplace transform of (A.8) we find

$$E(p) = p\bar{E}_R(p) = p\int_o^\infty E_R(t)e^{-pt}dt \tag{A.11}$$

which can be integrated numerically using the curve of $E_R(t)$ shown in Figure A-3. However, just as in the previous example, this is not readily done since $E_R(t)$ varies over a broad time interval*.

The more rapid method of determining a finite-element model consists of collocating the approximate series representation

$$E_s(t) = E_e + \sum_{i=1}^{n} E_i e^{-\frac{t}{\tau_i}} \tag{A.12}$$

where

$$E_e = \left[E(t)\right]_{t\to\infty}$$

with points on the experimental curve. For the particular example considered, it will be desirable to take as one condition ($t = t_1 = 0$)

$$E_s(0) = E(0) \tag{A.13}$$

and choose ten other points

$$t_i = 10^{(i-3)} \qquad (i = 2, 3, \cdots 11) \tag{A.14}$$

---

*However, when $\log E_R$ is essentially linear for several decades, such as shown in Figure A-3, then a first approximation to E(p) is

$$E(p) = \left[E_R(t)\right]_{t=\frac{\beta}{p}}$$

where β is defined in the footnote, page 7 . .

For the "relaxation times" τ_i , take $\tau_i = 2t_i$:

$$\tau_i = 2 \times 10^{(i-3)} \qquad (i = 1, 2, 3, \cdots 11) \tag{A.15}$$

Collocation at t = 0 and the values (A.14) yields the linear system:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0.007 & 0.607 & 0.951 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0.007 & 0.607 & 0.951 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0.007 & 0.607 & 0.951 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0.007 & 0.607 & 0.951 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0.007 & 0.607 & 0.951 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0.007 & 0.607 & 0.951 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.007 & 0.607 & 0.951 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.007 & 0.607 & 0.951 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.007 & 0.607 & 0.951 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.007 & 0.607
\end{pmatrix}
\begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \\ E_9 \\ E_{10} \\ E_{11} \end{pmatrix}
=
\begin{pmatrix}
E_R(0) - E_e \\
E_R(10^{-1}) - E_e \\
E_R(1) - E_e \\
E_R(10) - E_e \\
E_R(10^2) - E_e \\
E_R(10^3) - E_e \\
E_R(10^4) - E_e \\
E_R(10^5) - E_e \\
E_R(10^6) - E_e \\
E_R(10^7) - E_e \\
E_R(10^8) - E_e
\end{pmatrix} \tag{A.16}$$

Except for the top row, the elements in the matrix that are shown as zero or unity represent the exponential terms in the series (A.12) which, while not exactly zero or unity, are extremely close to these values. It is observed that if the 0.007 elements are neglected a triangular matrix results, which leads to a set of 1 x 1 equations for the E_i. These equations were easily solved to find

$$
\begin{aligned}
E_1 &= 194 \times 10^7 & E_7 &= 41 \times 10^7 \\
E_2 &= 283 \times 10^7 & E_8 &= 13.8 \times 10^7 \\
E_3 &= 554 \times 10^7 & E_9 &= 3.68 \times 10^7 \\
E_4 &= 602 \times 10^7 & E_{10} &= 0.79 \times 10^7 \\
E_5 &= 388 \times 10^7 & E_{11} &= 0.96 \times 10^7 \\
E_6 &= 156 \times 10^7 & E_e &= 2.24 \times 10^7
\end{aligned}
\qquad \text{(A.17)}
$$

which are sufficiently accurate values.

It only remains to substitute the coefficients (A.17) and relaxation times (A.15) into the operational modulus which is given by[6]

$$E(p) = E_e + \sum_{i=1}^{11} \frac{p\tau_i E_i}{p\tau_i + 1} \qquad \text{(A.18)}$$

This function is plotted in Figure A-3 for real, positive values of p.